More about Flight-Path-Angle Transitions in Optimal Airplane Climbs

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If the reciprocal of maximum L/D is treated as a small parameter ϵ in various optimal airplane climb problems, flight-path-angle transitions between vertical climbs or dives and "singular" climb arcs are found to follow a particular universal pattern investigated previously in connection with minimum-time climbs with negligible mass loss. The time scale of the transition is of order $\epsilon^{1/2}$, and universal formulas are given for the loss in payoff during transition due to the induced drag, the loss being of order $\epsilon^{3/2}$.

Introduction

In a recent paper,[1] the author showed that all flight-path-angle transitions to and from the maximum energy-rate path in the V-h plane, for the minimum-time climb problem with negligible mass loss, were described by a single "universal" nonlinear fourth-order differential system on an appropriately reduced time scale, at least if $\max(L/D)$ is large enough. The V-h paths were found to fare in to the maximum energy-rate path with a damped oscillation of damping ratio $1/\sqrt{2}$. It was tacitly expected that, if mass loss were taken into account, qualitatively similar behavior would occur.

It is the purpose of this paper to show that the transitions satisfy, on an appropriate time scale, the identical fourth-order system, not only when mass loss is taken into account in a minimum-time climb but also if the problem is changed, for example, to the maximum-altitude climb for given mass expenditure, time being free. The transition pattern, then, is, except for the exact time scale, independent not only of airplane characteristics but also of the particular optimization criterion, provided, of course, that the criterion is concerned with altitude and not with horizontal position.

Expressions also were derived in Ref. 1, using Green's theorem, for the time loss during transitions because of the induced drag. It will turn out that, although Green's theorem is no longer applicable, equivalent and now universal expressions are available for the loss in payoff during transitions in all optimal airplane climb problems.

Analysis of Transitions to or from a Singular Arc

The equations of motion are

$$\dot{h} = V \sin\gamma \tag{1a}$$

$$\dot{V} = (T-D)/m - g \sin\gamma \tag{1b}$$

$$V\dot{\gamma} = (L/m) - g \cos\gamma \tag{1c}$$

Thrust T will generally be a function $T(h,V)$. Drag D will be assumed to obey the parabolic formula:

$$D(h,V,L) = D_0(h,V) + [\epsilon^2 L^2 / D_0(h,V)] \tag{2}$$

where

$$\epsilon = [2(L/D)_{\max}]^{-1} \tag{3}$$

ϵ being, in general, dependent on Mach number; ϵ will be treated as a small parameter.

The Hamiltonian to be maximized in any optimal climb problem has the form

$$H = \lambda_h V \sin\gamma + \lambda_V \left(\frac{T - D_0 - (\epsilon^2 L^2 / D_0)}{m} - g \sin\gamma\right) + \frac{\lambda_\gamma}{V}\left(\frac{L}{m} - g\cos\gamma\right) + (\text{terms without } L,\gamma) \tag{4}$$

where the terms without L, γ can include $\lambda_m \dot{m}$ and, in the case of minimum-time climb, an additive term -1.

The optimal control (the lift L) is obtained by maximizing H as

$$L = \lambda_\gamma D_0 / 2\epsilon^2 V \lambda_V \tag{5}$$

and λ_γ has a time rate

$$\dot{\lambda}_\gamma = -\lambda_\gamma (g/V) \sin\gamma - S\cos\gamma \tag{6}$$

where

$$S = V\lambda_h - g\lambda_V \tag{7}$$

which is the switch function for the control $\sin\gamma$ in the case $\epsilon = 0$, when induced drag is neglected, so that $\lambda_\gamma = 0$. In this case, the optimal paths are made up of vertical climbs or dives, $\sin\gamma = \pm 1$, and of "singular" arcs on which S remains zero.

Now, assuming that, as in the minimum-time climb problem with negligible mass loss, a singular arc is always a "first-order" singular arc, $\ddot{S}$ is expressible (if $\epsilon = 0$) in the form

$$\ddot{S} = k_1(h,V,m,\lambda_h,\lambda_V,\ldots)(\sin\gamma - \sin\gamma_S) \tag{8}$$

where $k_1 > 0$ (the "generalized Legendre-Clebsch" condition) and where γ_S denotes the value of γ on the singular arc, varying generally with time.

For small positive ϵ, it turns out that Eq. (8) describes the dominant part of $\ddot{S}$. But, assuming that $|L| \gg mg$ for the main part of a transition,

$$\dot{\gamma} \cong L/mV = k_2 \lambda_\gamma / \epsilon^2 \tag{9}$$

where

$$k_2 = D_0 / 2mV^2 \lambda_V \tag{10}$$

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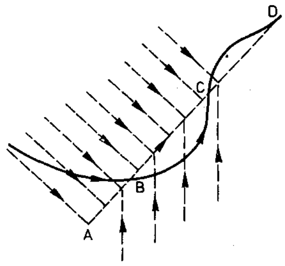

Fig. 1 Optimal trajectories of singular problem near arrival.

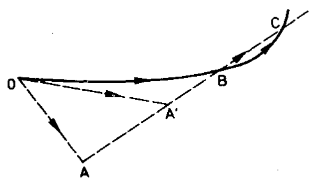

Fig. 2 Optimal trajectories of singular problem near arrival along a constraint.

and hence

$$\ddot{\gamma} = -(k_2/\epsilon^2) S \cos\gamma + (\text{terms with } \dot{\gamma}) \tag{11}$$

in which the relatively slow rate of change of k_2 has been ignored.

We now introduce, in the neighborhood of any corner junction with the singular arc for the case $\epsilon = 0$, the rescaled variables:

$$\tau = (t/\sqrt{\epsilon})(k_1 k_2)^{1/4}, \quad \sigma = (S/\epsilon)(k_2/k_1)^{1/2} \tag{12}$$

and obtain, just as in Ref. 1,

$$\frac{d^2\gamma}{d\tau^2} = -\sigma \cos\gamma, \qquad \frac{d^2\sigma}{d\tau^2} = \sin\gamma - \sin\gamma_S \tag{13}$$

in which terms of order $\epsilon^{1/2}$ have been neglected.

Near $\gamma = \gamma_S$, the transition equations (13) reduce to

$$\frac{d^4(\gamma - \gamma_S)}{d\tau^4} \cong -(\gamma - \gamma_S)\cos^2\gamma_S \tag{14}$$

and, near $\gamma = \pm\pi/2$, $\sigma \sim \pm(\tau^2/2)(1 \mp \sin\gamma_S)$ for large $|\tau|$, so that

$$\frac{\pi}{2} \mp \gamma \sim A\exp\left[\frac{\tau^2}{2}\sqrt{\frac{1 \mp \sin\gamma_S}{2}}\right] + B\exp\left[-\frac{\tau^2}{2}\sqrt{\frac{1 \mp \sin\gamma_S}{2}}\right] \tag{15}$$

which can approach zero as $|\tau| \to \infty$ only if $A = 0$. Since Eqs. (13) are invariant under time reversal, the transition pattern is the same whether arriving at or departing from the singular arc. The pattern is described by that solution of Eqs. (13) for which $A = 0$ near $\gamma = \pm\pi/2$ and which approaches the singular arc in forward or backward time; i.e., for which the modes of the linear system (14) corresponding to poles in the right or left half-plane, respectively, are suppressed (see Ref. 1, Fig. 2). The damping ratio of approach, in forward or backward time, to the singular arc is clearly $1/\sqrt{2}$. The pattern can be obtained numerically, for any particular γ_S, by adjusting the phase of this damped oscillation, at a given small amplitude (e.g., 0.01 rad) of its exponential envelope, so that integration of Eqs. (13) does yield $A = 0$. Since $d\gamma/d\tau$ is of order 1 for the main part of the transition, it follows from Eqs. (9) and (12) that L is here of order $\epsilon^{-1/2}$, thus justifying the assumption $|L| \gg mg$.

A similar analysis would apply to a junction of a state-constrained path $h = 0$ onto a singular arc climb. Here the phase of the forward-damped oscillating approach to the singular arc is adjusted so that backward integration of Eqs. (13) yields $d\gamma/d\tau = 0$ simultaneously with $\gamma = 0$.

Loss in Payoff During Transitions ($\epsilon > 0$)

If J denotes the payoff whose maximization is desired, the loss $-\Delta J$ is the sum of two parts: 1) the direct loss due to the induced drag $\epsilon^2 L^2/D_0$, and 2) the loss, when $\epsilon = 0$, due to following the transition path instead of the cornered path. The first loss is given by

$$\Delta_1 J = \lambda_V \Delta V$$

where

$$\Delta V = -\int \frac{\epsilon^2 L^2}{m D_0} dt$$

and λ_V is evaluated at (or near) the corner. Thus,

$$-\Delta_1 J = \frac{k_1^{1/4}\epsilon^{3/2}}{2k_2^{3/4}} \int_{-\infty}^{\infty} \left(\frac{d\gamma}{d\tau}\right)^2 d\tau \tag{16}$$

The second loss requires a more detailed analysis. In Ref. 1, where there were only two state variables (for $\epsilon = 0$), $\Delta_2 J$ was obtained by Green's theorem. A different analysis is necessary here. Suppose, for example, that the junction is from an arc $\sin\gamma = -1$ onto a singular arc:

$$\Delta_2 J = \int \frac{\partial J(x)}{\partial x} [\dot{x}(\gamma) - \dot{x}(\gamma_{opt})] dt$$

where x denotes the state $(h, V, \ldots)$, and $J(x)$ denotes the maximum J for the singular problem ($\epsilon = 0$, $\lambda_\gamma = 0$) with initial state x. Thus,

$$\Delta_2 J = \int S^*(x)(\sin\gamma - \sin\gamma_{opt}) dt$$

where $S^*(x)$ is the value of S at state x along the optimal cornered path. Now, near a junction corner A, an important state variable is

$$\dot{S} = -\int_t^{\infty} k_1(\sin\gamma - \sin\gamma_S) dt$$

and, since $\ddot{S}$ is constant, equal to $k_1(\mp 1 - \sin\gamma_S)$ according to whether the path (for $\epsilon = 0$) approaches the singular arc from above or below in Fig. 1, and since $S^* = \dot{S} = 0$ on the singular arc,

$$S^* = \mp \dot{S}^2 / [2k_1(1 \pm \sin\gamma_S)]$$

Thus,

$$\Delta_2 J = -\frac{1}{2k_1(1+\sin\gamma_S)}\left(\int_{-\infty}^{t_B} + \int_{t_C}^{t_D} + \cdots\right)\dot{S}^2(\sin\gamma + 1) dt$$
$$-\frac{1}{2k_1(1-\sin\gamma_S)}\left(\int_{t_B}^{t_C} + \cdots\right)\dot{S}^2(1 - \sin\gamma) dt$$

But

$$\int_{t_B}^{t_C} \frac{\dot{S}^2(1-\sin\gamma)}{1-\sin\gamma_S} dt$$
$$= \int_{t_B}^{t_C}\left[\frac{\dot{S}^2(1+\sin\gamma)}{1+\sin\gamma_S} - \frac{2\dot{S}^2(\sin\gamma - \sin\gamma_S)}{1-\sin^2\gamma_S}\right] dt$$

and

$$\int_{t_B}^{t_C} \dot{S}^2(\sin\gamma - \sin\gamma_S) dt = \frac{1}{k_1}\int_{\dot{S}_B}^{\dot{S}_C} \dot{S}^2 d\dot{S} = 0$$

since $\dot{S}$, like S, vanishes along the singular arc. Thus,

$$\int_{t_B}^{t_C} \frac{\dot{S}^2 (1-\sin\gamma)}{1-\sin\gamma_S} \mathrm{d}t = \int_{t_B}^{t_C} \frac{\dot{S}^2 (1+\sin\gamma)}{1+\sin\gamma_S} \mathrm{d}t$$

The same is true for $\int_{t_D}^{t_E}$, etc., and we now obtain

$$\Delta_2 J = -\frac{1}{2k_1}\int_{-\infty}^{\infty} \dot{S}^2 \frac{1+\sin\gamma}{1+\sin\gamma_S}\mathrm{d}t$$

and hence, as in Ref. 1,

$$-\Delta_2 J = \frac{k_1^{1/4}\epsilon^{3/2}}{2k_2^{3/4}}\int_{-\infty}^{\infty} \frac{1+\sin\gamma}{1+\sin\gamma_S}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\right)^2 \mathrm{d}\tau \tag{17}$$

For transitions from a vertical climb $\sin\gamma = +1$ onto a singular arc, the same argument leads to replacement of $(1+\sin\gamma)/(1+\sin\gamma_S)$ in Eq. (17) by $(1-\sin\gamma)/(1-\sin\gamma_S)$, as in Ref. 1. In the case of a transition from the state-constrained path $h=0$ onto a singular arc, comparison with the path starting along $0A'$ ($h=0$) instead of along $0A$ in Fig. 2 yields

$$[\Delta_2 J]_0^B = -\frac{1}{2k_1^2(1+\sin\gamma_S)} \times \int_0^{\dot{S}_0}\left[\frac{\dot{S}^2(1+\sin\gamma)}{\sin\gamma - \sin\gamma_S} - \frac{\dot{S}^2}{(-\sin\gamma_S)}\right]\mathrm{d}\dot{S}$$

$$= -\frac{1}{2k_1}\int_{t_0}^{t_B} \dot{S}^2 \frac{\sin\gamma}{\sin\gamma_S}\mathrm{d}t$$

whereas $[\Delta_2 J]_B^C$, $[\Delta_2 J]_{C'}^D, \cdots$, also are expressible with the same integrand. Hence, $(1+\sin\gamma)/(1+\sin\gamma_S)$ in Eq. (17) is replaced by $\sin\gamma/\sin\gamma_S$, again as in Ref. 1.

The loss in payoff during transitions from the singular arc to a vertical path, $\sin\gamma = \pm 1$, is given by identical formulas. For example, if the final transition is to $\gamma = +\pi/2$ and if the optimization problem has final altitude h_f free, as is the case for the problem of maximum altitude for prescribed time t_f and final velocity V_f, the optimal paths ($\epsilon = 0$) near the corner have the form shown in Fig. 3. Here $\gamma_{\text{opt}} = +\pi/2$ both above and below the singular arc, so that

$$S^* = +\frac{\dot{S}^2}{2k_1(1-\sin\gamma_S)}$$

and

$$-\Delta_2 J = \frac{k_1^{1/4}\epsilon^{3/2}}{2k_2^{3/4}}\int_{-\infty}^{\infty}\frac{1-\sin\gamma}{1-\sin\gamma_S}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\right)^2\mathrm{d}\tau$$

just as for paths arriving at a singular arc from a vertical climb $\gamma = \pi/2$.

If, on the other hand, the problem is the companion problem of minimizing time to a fixed final state, all of the adjoint variables λ are multiplied by $\partial t_f/\partial h_f$. Thus the factor $k_1^{1/4}/k_2^{3/4}$, being homogeneous of first degree in the λ's, is multiplied by the same factor, and $-\Delta_2 J$ is thus again the correct $\Delta_2 t$.

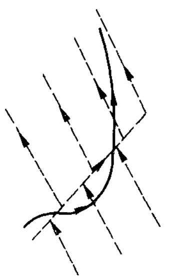

Fig. 3 Optimal trajectories of singular problem near departure corner.

Table 1 Transition to $\gamma = \pi/2$

γ_S(rad)	$-\frac{2k_2^{3/4}}{\epsilon^{3/2}k_1^{1/4}}(\Delta_1 J + \Delta_2 J)$	$\max\left\lvert\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right\rvert$
−0.2[a]	1.09	0.63
0	0.83	0.53
0.2	0.61	0.46
0.4	0.37	0.42
0.6	0.28	0.27
0.8	0.20	0.15

[a] Junction of $-\gamma_S$ with $\gamma = \pi/2$ gives same values as junction of $+\gamma_S$ with $\gamma = -\pi/2$.

As pointed out in Ref. 1, $\Delta_2 t$ is in every case equal to $3\Delta_1 t$, since rescaling of the independent variable in Eqs. (13) by a factor R leads to $\Delta_1 J \to R^{-1}\Delta_1 J$ and $\Delta_2 J \to R^3 \Delta_2 J$. Since $-\Delta_1 J - \Delta_2 J$ must be minimized by $R=1$, the 3:1 ratio is established and has been verified numerically for several values of γ_S. Table 1 shows a dimensionless loss in payoff as well as the maximum $|\mathrm{d}\gamma/\mathrm{d}\tau|$ during transition for various values of γ_S.

Other Transitions

Consider now the transitions between an initial vertical dive $\gamma = -\pi/2$ and the state constraint $h=0$. Along the state constraint, λ_h is augmented by a negative quantity $\mu(t)$ such that h remains zero. In the limiting case $\epsilon = 0$, this requires that $\sin\gamma = 0$ maximize H (with $\epsilon = \lambda_\gamma = 0$) and hence that S remain zero along the constraint. But now $\dot{S}$ assumes the positive value $k_3 = -V\mu(t_1)$ immediately prior to arrival at the constraint at time t_1. For $\epsilon > 0$, it will turn out that S is a positive quantity of order $\epsilon^{2/3}$ on arrival at the constraint. In fact, γ again satisfies $\ddot{\gamma} \cong -(k_2/\epsilon^2)S\cos\gamma$.

Introducing the rescaled variables

$$\tau = -[(k_2 k_3)^{1/3}/\epsilon^{2/3}](t - t_1) \tag{18a}$$

$$\sigma = [k_2^{1/3}/(k_3\epsilon)^{2/3}]S \tag{18b}$$

the rescaled time τ being measured backward from arrival at the constraint, we obtain

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau^2} = -\sigma\cos\gamma, \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = -1 \tag{19}$$

with $\gamma = \mathrm{d}\gamma/\mathrm{d}\tau = 0$ at $\tau = 0$ (since the optimal $\mathrm{d}\gamma/\mathrm{d}\tau$ has to be continuous at $\tau = 0$) and $\gamma \to -\pi/2$ as $\tau \to \infty$. Near $\gamma = -\pi/2$, τ is large, and $(\mathrm{d}^2/\mathrm{d}\tau^2)[(\pi/2)+\gamma] \cong \tau[(\pi/2)+\gamma]$. The boundary condition $\gamma \to -\pi/2$ as $\tau \to \infty$ thus uniquely determines $\sigma(0)$; it was found by numerical iteration to be 1.202702.

The loss in payoff is again made up of two parts:

$$-\Delta_1 J = \lambda_V \int \frac{\epsilon^2 L^2}{mD_0}\mathrm{d}t = \frac{\epsilon^{4/3}k_3^{1/3}}{2k_2^{2/3}}\int_0^{\infty}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right)^2\mathrm{d}\tau \tag{20}$$

and

$$-\Delta_2 J = -\int S^*(h)(1+\sin\gamma)\mathrm{d}t$$

where

$$S^*(h) = -k_3\frac{h}{V} = -k_3\frac{\epsilon^{2/3}}{(k_2 k_3)^{1/3}}\int_0^{\tau}[-\sin\gamma(\tau')]\mathrm{d}\tau'$$

so that

$$-\Delta_2 J = \frac{\epsilon^{4/3}k_3^{1/3}}{k_2^{2/3}}\int_0^{\infty}[1+\sin\gamma(\tau)]\int_0^{\tau}[-\sin\gamma(\tau')]\mathrm{d}\tau'\mathrm{d}\tau \tag{21}$$

again as in Ref. 2. Numerically we found $-\Delta_1 J = -2\Delta_2 J = 0.404(k_3^{1/3}/k_2^{2/3})\ \epsilon^{4/3}$, the 2:1 ratio being a consequence of the fact that rescaling of the independent variable in Eqs. (19) now yields $\Delta_1 J \to R^{-1}\Delta_1 t$, $\Delta_2 J \to R^2 \Delta_2 J$.

Consider finally the case when initial and/or final γ is prescribed to be zero. A transition to final $\gamma_f = 0$ is governed by

$$\ddot{\gamma} = -(k_4/2\epsilon^2)\cos\gamma \tag{22}$$

where

$$k_4 = SD_0/mV^2\lambda_V \tag{23}$$

and S is now positive. Introducing the rescaled time

$$\tau = \sqrt{k_4}/\epsilon \tag{24}$$

we obtain

$$\frac{d^2\gamma}{d\tau^2} = -\frac{1}{2}\cos\gamma \tag{25}$$

and hence

$$\frac{d\gamma}{d\tau} = -\sqrt{1-\sin\gamma} \tag{26}$$

and

$$\tau_f - \tau = \sqrt{2}\ ln \frac{\tan(\pi/8)}{\tan[(\pi/8)-(\gamma/4)]} \tag{27}$$

The loss in payoff is $-\Delta_1 J - \Delta_2 J$, where now

$$\Delta_1 J = -\lambda_V \int \frac{\epsilon^2 L^2}{mD_0} dt$$

and

$$\Delta_2 J = S\int(\sin\gamma - 1)\, d\tau$$

Here, as easily anticipated,

$$\Delta_1 J = \Delta_2 J \tag{28}$$

and

$$-\Delta_2 J = (2\epsilon S/\sqrt{k_4})(\sqrt{2}-1) \tag{29}$$

A similar analysis applies to an initial constraint $\gamma_0 = 0$. Transition to a vertical dive $\gamma = -\pi/2$ leads again to Eq. (25) after a time-scale change given by Eq. (23), with k_4 now equal to $|S|D_0/mV^2\lambda_V$, S being negative. The transition pattern is

$$\tau = \sqrt{2}\ \ln \frac{\tan(\pi/8)}{\tan[(\pi/8)+(\gamma/4)]} \tag{30}$$

and

$$-\Delta_1 J = -\Delta_2 J = (2\epsilon|S|/\sqrt{k_4})(\sqrt{2}-1) \tag{31}$$

Concluding Remarks

The most gradual transitions have been shown to be those to or from the singular arc, the unit of transition time ($\tau = 1$) being of order $\epsilon^{1/2}$ rather than $\epsilon^{2/3}$ or ϵ. The corresponding maximum lift during transition is thus of order $\epsilon^{-1/2}$ rather than $\epsilon^{-2/3}$ or ϵ^{-1}, and the corresponding loss in payoff is of order $\epsilon^{3/2}$ rather than $\epsilon^{4/3}$ or ϵ. If, as is customary, indirect drag with $L = mg$ is included in $D_0(h,V)$, the resulting path and payoff corrections are of order only ϵ^2. The transition payoff losses quoted are thus correct for small ϵ.

The mathematical validity of the entire analysis depends on the relative constancy of the states $h, V, \cdots$ during the unit of transition time, $\tau = 1$, e.g., during $\Delta t = \sqrt{\epsilon}/(k_1 k_2)^{1/2}$, for the transitions to or from a singular arc. In Ref. 1, this requirement was investigated for a minimum-time climb using a simplified aerodynamic model: $T = \text{const}$, $D_0 = KV^2 e^{-h/H}$. It was concluded that the requirement was violated at transonic or supersonic speeds, especially for an airplane at high altitude. This does not rule out the possibility that the losses in payoff quoted in Ref. 1 and in this paper might be numerically good approximations. Ardema[2] used a different analysis for the minimum-time climb problem according to which the energy state, $E = (V^2/2) + gh$, was treated as a slow variable by comparison with h and γ. His numerical results on minimum time were encouraging, notwithstanding that the time taken for significant changes in h was comparable with that for significant changes in E.

References

[1] Breakwell, J.V., "Optimal Flight-Path-Angle Transitions in Minimum Time Airplane Climbs," *Journal of Aircraft*, Vol. 14, Aug. 1977, pp. 782-786.

[2] Ardema, M., "Singular Perturbations in Optimal Control and Solution of the Aircraft Minimum Time-to-Climb Problem by the Method of Matched Asymptotic Expansions," *International Meeting on Optimization Problems in Engineering and Economics*, Naples, Italy, Dec. 1974.